Stability of Non-Viscous Incompressible Dusty Fluid Flow in Rotating Annulus

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Abstract

This paper theoretically investigates the stability of circumferential laminar dusty flow between two coaxial rotating circular cylinders. The reaction of the system to infinitesimally small axisymmetric perturbations is examined under the assumptions that relaxation time is small and the sedimentation velocity is negligible. The results include the stability of oscillatory modes, stability of non-oscillatory modes under the condition $DN_0 < 0$ *and non-existence of neutral modes.*

Introduction

The problem of the stability of circumferential laminar flows between two coaxial circular cylinders has a vast history. Rayleigh[1] provided the necessary and sufficient criterion of stability on physical grounds and Synge[2] gave the first analytical proof of Rayleigh's criterion of stability. Saffman[3] gave a formulation of the problem of linearised stability of a plane parallel flow of a dusty gas. The effect of dust is described by two parameters: the concentration of dust and a relaxation time. A number of physical situations are associated with the flow of dusty flow between two rotating cylinders. Such type of flow has been discussed by Greenspan[4], Ungarish[5]. Gireesha and Madhura[6] examined unsteady flow of a dusty fluid through porous media between annulus of two hexagonal channels. N. Dutta etl.[7] discussed the Pulsatile flow and heat transfer of dusty fluid through an infinitely long annular pipe. Saha etl.[8] investigated natural convection of dusty nanofluids in an annulus. The present problem to the best of our knowledge has not been discussed so far. The basic assumptions that we make in simplifying the model are:

(i) Gap between two coaxial cylinders is small as compared to their mean radii.
(ii) Perturbations are axisymmetric in nature.
(iii) Relaxation time is small.
(iv) Number density depends upon *r*.
(v) Velocity of sedimentation is negligible.

These assumptions are examined below with proper justification for each.

The narrow gap approximation is made here for the sake of simplifying the basic flow velocity and also the linearised perturbation equations. This approximation idealises the situation when the common length of the cylinder is large as compared to their radii and is usually met within experiments. We shall restrict our attention to the case of axisymmetric perturbations. This is an important restriction for if non-axisymmetric perturbations are included, new mechanism of instability may become possible. The relaxation timeτmeasures the rate at which the velocity of dust particles adjusts to change in the gas velocity and it depends upon the size of the individual particles. Therefore, for fine dust particle, τ will be small as compared to a characteristic time scale associated with the flows. It has been established by Saffman and others that the fine dust destabilizes and the coarse dust stabilizes the flow and thus a situation predicted to be stable in the presence of fine dust particles is expected to remain stable even in the presence of coarse dust particles. Moreover, we are assuming that the angular velocity of the cell containing dusty fluid is not too high to produce sedimentation. In view of this, the velocity of sedimentation is small as compared with characteristic velocity of the flow and therefore can be neglected.

Formulation of the problem

Consider a non-viscous incompressible laminar dusty Couette flow between two coaxial circular cylinders of radii *a* and *b* (*b*>*a*) of infinite length with cylindrical polar coordinates (r, θ, z) , where the axis of the annulus is assumed to be along *z*-axis and the dusty flow is in the region $a \leq r \leq b$. The motion of the fluid is due to the rotation of circular cylinders.

Governing equation of motion

The equations of motion of an incompressible, non-viscous dusty Couette flow in cylindrical polar coordinates are:

For Clean Fluid:

The equations of momentum are

$$\rho\left[\frac{\partial u_r}{\partial t} + (u.\nabla)u_r - \frac{{u_\theta}^2}{r}\right] = -\frac{\partial p}{\partial r} + K^*N(v_r - u_r) \qquad (1)$$

$$\rho\left[\frac{\partial u_\theta}{\partial t} + (u.\nabla)u_\theta + \frac{u_r u_\theta}{r}\right] = -\frac{\partial p}{r\partial \theta} + K^*N(v_\theta - u_\theta) \qquad (2)$$

$$\rho\left[\frac{\partial u_z}{\partial t} + (u.\nabla)u_z\right] = -\frac{\partial p}{\partial z} + K^*N(v_z - u_z) \qquad (3)$$

Equation of Continuity

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \qquad (4)$$

Equation of incompressibility

$$\frac{\partial \rho}{\partial t} + u_r\frac{\partial \rho}{\partial r} + \frac{u_\theta}{r}\frac{\partial \rho}{\partial \theta} + u_z\frac{\partial \rho}{\partial z} = 0 \qquad (5)$$

For Dust Particles

$$mN\left[\frac{\partial v_r}{\partial t} + (v.\nabla)v_r - \frac{{v_\theta}^2}{r}\right] = -mNg + K^*N(u_r - v_r) \qquad (6)$$

$$mN\left[\frac{\partial v_\theta}{\partial t} + (v.\nabla)v_\theta - \frac{v_r v_\theta}{r}\right] = K^*N(u_\theta - v_\theta) \qquad (7)$$

$$mN\left[\frac{\partial v_z}{\partial t} + (v.\nabla)v_z\right] = K^*N(u_z - v_z) \qquad (8)$$

Equation of incompressibility

$$\frac{\partial N}{\partial t} + v_r\frac{\partial N}{\partial r} + \frac{v_\theta}{r}\frac{\partial N}{\partial \theta} + v_z\frac{\partial N}{\partial z} = 0 \qquad (9)$$

Where

$$u.\nabla \equiv u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

$$v.\nabla \equiv v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z}$$

Here *t* is the time, *p* the pressure, $K^* = 6\pi a\mu$ the Stoke's resistance coefficient, *r* being the radius of dust particles assumed to be spherical, *m* and N respectively the mass and the number density of dust particles, *g* the magnitude of the acceleration due to gravity and ***u*** and ***v*** are respectively the velocity of clean fluid and velocity of dust particles.

Basic State of the motion

The above equations clearly allow the time independent solution (called the basic state) as

$$\left.\begin{aligned} u_r = 0 = u_z \text{ and } u_\theta = U(r) \\ v_r = 0 = v_z \text{ and } v_\theta = U(r) \\ \frac{\partial p}{\partial r} = \frac{\rho U^2}{r},\ g = \frac{U^2}{r} \end{aligned}\right\} \qquad (10)$$

Perturbation Equations

Let $\rho', p', (u_r', u_\theta', u_z')$, and (v_r', v_θ', v_z') denote respectively the perturbations in density ρ, pressure *p*, velocity of clean fluid ***u*** and velocity of dust particles ***v***. Linearizing the equations in perturbations and analysing the perturbations into normal modes of the form

$$f(r)\exp i[st + kz] \qquad (11)$$

Where $s = s_r + is_i$ is a constant which is complex in general and *k* is the real wave number. After dropping the primes (′) and using the transformations

$$u_r = is\xi_r, \quad u_\theta = is\xi_\theta - \left(\frac{dU}{dr} - \frac{U}{r}\right)\xi_r \text{ and } u_z = is\xi_z$$

and eliminating various physical quantities in favour of ξ_r, we have

$$s^2(DD_* - k^2)\xi_r + k^2\Phi(r)\xi_r - \left[\frac{k^2K^* g\tau s D N_0}{\rho_0\left(s + i\frac{DN_0}{N_0}g\tau\right)}\right]\xi_r = 0 \qquad (12)$$

with the boundary conditions

$\xi_r = 0$ at $r = a$ and $r = b$

Where $\Omega = \frac{U}{r}, D \equiv \frac{d}{dr}, D_* \equiv \frac{d}{dr} + \frac{1}{r}, \tau = \frac{m}{K^*}$ and $\Phi(r) = 2r\Omega\frac{d\Omega}{dr} + 4\Omega^2$ is the Rayleigh's discriminant.

Now multiplying equation (12) by $r\xi_r^*$ and integrating over the range of r, we get

$$s^2\int_a^b r\left|\frac{d\xi_r}{dr} + \frac{\xi_r}{r}\right|^2 dr + k^2 s^2\int_a^b r|\xi_r|^2 dr - k^2\int_a^b r\Phi(r)|\xi_r|^2 dr$$

$$+k^2\int_a^b r\left[\frac{K^* g\tau s D N_0}{\rho_0(s + i\frac{DN_0}{N_0}g\tau}\right]|\xi_r|^2 dr = 0 \qquad (13)$$

Separating real and imaginary parts of equation (13), the real and imaginary parts respectively are

$$(s_r{}^2 - s_i{}^2)\int_a^b r\left|\frac{d\xi_r}{dr} + \frac{\xi_r}{r}\right|^2 dr + k^2(s_r{}^2 - s_i{}^2)\int_a^b r|\xi_r|^2 dr - k^2\int_a^b r\Phi(r)|\xi_r|^2 dr$$
$$+k^2\int_a^b \frac{K^* g\tau|s|^2 DN_0}{\rho_0\left|s+i\frac{DN_0}{N_0}g\tau\right|^2} r|\xi_r|^2 dr + s_i k^2\int_a^b \frac{K^*(DN_0 g\tau)^2}{N_0\rho_0\left|s+i\frac{DN_0}{N_0}g\right|^2} r|\xi_r|^2 dr = 0 \tag{14}$$

and

$$2s_r s_i\int_a^b r\left|\frac{d\xi_r}{dr} + \frac{\xi_r}{r}\right|^2 dr + 2k^2 s_r s_i\int_a^b r|\xi_r|^2 dr$$
$$-k^2 s_r\int_a^b \frac{K^*(g\tau DN_0)^2}{N_0\rho_0\left|s+i\frac{DN_0}{N_0}g\tau\right|^2} r|\xi_r|^2 dr = 0 \tag{15}$$

Let $I_1 = 2\int_a^b r\left|\frac{d\xi_r}{dr} + \frac{\xi_r}{r}\right|^2 dr$, $I_2 = 2k^2\int_a^b r|\xi_r|^2 dr$, and

$$I_3 = k^2\int_a^b \frac{K^*(g\tau DN_0)^2}{N_0\rho_0\left|s+i\frac{DN_0}{N_0}g\tau\right|^2} r|\xi_r|^2 dr$$

Then equation (15) can be written as

$$s_r(s_i I_1 + s_i I_2 - I_3) = 0 \tag{16}$$

We now prove the following theorems:

Theorem I: The oscillatory modes, if exist, are stable.

Proof: For oscillatory modes $s_r \neq 0$. Let the modes be oscillatory, then from equation (16)

$$s_i(I_1 + I_2) - I_3 = 0 \tag{17}$$

Since I_1, I_2 and I_3 are all positive definite integrals, then from equation (17) it clearly ensures that $s_i > 0$, so that the oscillatory modes, if exist, are stable.

Theorem II: If Ω is an increasing function of *r*, then non-oscillatory modes are stable when

$$DN_0 < 0.$$

Proof: For non-oscillatory modes, $s_r = 0$. Let the modes be unstable so that $s_i < 0$. Then equation (14) becomes

$$(-s_i{}^2)\int_a^b r\left|\frac{d\xi_r}{dr} + \frac{\xi_r}{r}\right|^2 dr + k^2(-s_i{}^2)\int_a^b r|\xi_r|^2 dr - k^2\int_a^b r\Phi(r)|\xi_r|^2 dr$$
$$+k^2\int_a^b \frac{K^* g\tau|s_i|^2 DN_0}{\rho_0\left|s_i+\frac{DN_0}{N_0}g\tau\right|^2} r|\xi_r|^2 dr + s_i k^2\int_a^b \frac{K^*(DN_0 g\tau)^2}{N_0\rho_0\left|s_i+\frac{DN_0}{N_0}g\right|^2} r|\xi_r|^2 dr = 0 \tag{18}$$

Since $\frac{d\Omega}{dr} > 0$ and $\Phi(r) = 2r\Omega\frac{d\Omega}{dr} + 4\Omega^2$

so that $\Phi(r) > 0$.Now the equation (18) is inconsistent if $DN_0 < 0$.Thus above equation will be inconsistent if $s_i < 0$. Hence non-oscillatory modes are stable.

Theorem III: If Ω is a decreasing function of *r*, then non-oscillatory modes are stable if
$DN_0 < 0$ and $0 < \Phi(r) < 4\Omega^2$.

Proof: We know that $s_r = 0$ for non-oscillatory modes and let the modes be unstable so that

$s_i < 0$. Since Ω is a decreasing function of *r* then $\frac{d\Omega}{dr} < 0$, therefore, $0 < \Phi(r) < 4\Omega^2$. The equation (18) will be inconsistent if $DN_0 < 0$ and $0 < \Phi(r) < 4\Omega^2$. Hence non-oscillatory modes will be stable if $DN_0 < 0$ and $0 < \Phi(r) < 4\Omega^2$.

Theorem IV: Neutral modes cannot exist.

Proof: Dividing equation (13) by *s* and separating imaginary part, we have

$$s_i[\int_a^b \left|\frac{d\xi_r}{dr} + \frac{\xi_r}{r}\right|^2 dr + \frac{k^2}{|s|^2}\int_a^b |\xi_r|^2 dr + \frac{k^2}{|s|^2}\int_a^b r\Phi(r)|\xi_r|^2 dr - k^2\int_a^b \frac{K^* g\tau DN_0}{\rho_0\left|s + i\frac{DN_0}{N_0}g\tau\right|^2} r|\xi_r|^2 dr] - k^2\int_a^b \frac{K^*(g\tau DN_0)^2}{\rho_0 N_0\left|s + i\frac{DN_0}{N_0}g\tau\right|^2} r|\xi_r|^2 dr = 0 \quad (19)$$

Let the modes be neutral so that $s_i = 0$ (here s_r cannot be zero because there is division by *s*), then above equation implies that

$$\int_a^b \frac{K^*(g\tau DN_0)^2}{\rho_0 N_0\left|s + i\frac{DN_0}{N_0}g\tau\right|^2} r|\xi_r|^2 dr = 0$$

Which is not possible. It follows that s_i cannot be zero. Hence neutral modes cannot exist.

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